ON THE STABILITY OF STATIONARY MOTIONS OF RIGID BODIES WITH CAVITIES CONTAINING FLUID

(OB USTOICHIVOSTI USTANOVIVSHIKHSIA DVIZHENII TVERDYKH TEL S POLOSTIAMI, NAPOLNENYMI ZHIDKOST'IU)

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V.V. RUMIANTSEV
(Moscow)

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This paper deals with the stability of steady motions of rigid bodies with cavities filled, partially or completely, with homogeneous incompressible fluid. The investigation is based on the concepts developed by Liapunov [1,2] in the theory of stability of the equilibrium configurations of rotating fluids. It is shown that the problem of stability of uniform rotation of a rigid body with a cavity containing fluid reduces to the investigation of the conditions of minimum of a certain expression W. If the fluid fills the cavity completely, W becomes a function of a finite number of variables; if the fluid fills the cavity partially, W represents a functional depending on the coordinates of the rigid body and on the form of the fluid.

The application of the obtained theorems is illustrated by solving two problems of stability of steady motions of rigid bodies with cavities containing fluid in the following cases: (1) A free body in the Newtonian gravitational field with a fixed center; (2) A free body moving around a fixed point in a homogeneous gravitational field. These two problems are of considerable interest.

1. We shall consider a rigid body with a simply-connected cavity of arbitrary shape filled with a homogeneous incompressible ideal fluid. We shall discuss simultaneously the case of the cavity filled completely and the case of partial filling with the fluid having a free surface acted upon by a constant hydrostatic pressure. The particles of the fluid which are in contact with the walls of the cavity have the normal components of velocities equal to the normal components of velocities of the corresponding points of the walls.

We shall assume that the body is subjected to scleronomic non-releasing constraints; the prescribed forces acting on the body and the mass forces acting on the fluid particles have the force functions U_1 and U_2 which do not depend on time explicitly. We shall also assume that the motion of the body is continuous in time, and the fluid moves as a continuum, i.e. the coordinates of the particles of the fluid are continuous functions of time and their initial positions.

With the above assumptions, the differential equations of motion admit the integral of energy [3]

$$T + V = h \tag{1.1}$$

where T is the kinetic energy of the system moving with respect to a fixed coordinate system $O\xi\eta\zeta$, V is the potential energy of the forces acting on the system, h is the constant of integration.

The position of the rigid body, with respect to the coordinate system $O\xi\eta\zeta$ will be determined by its Lagrangean coordinates $q_1, \ldots, q_n$ $(n \leqslant 6)$. The potential energy V will be, in general, a function of the coordinates $q_1, \ldots, q_r$ $(r \leqslant n)$ and of the configuration of the fluid. In the case when the fluid fills the cavity completely, the potential energy of the system is the function $V(q_1, \ldots, q_r)$.

We shall consider that the imposed constraints allow for rotation of the whole system as one rigid body around a fixed axis, and that the forces acting on the system produce no moment with respect to this axis. Consequently, the potential energy of the system V does not depend on the angle q_n of rotation around this axis. Under these conditions, the integral of areas exists for the plane perpendicular to the axis of rotation [3]. Assuming that the axis of rotation coincides with the coordinate axis $O\zeta$ belonging to the fixed coordinate system, we write the integral of areas in the form

$$G_\zeta = \text{const} = k \tag{1.2}$$

where G_ζ is the projection of the angular momentum on the ζ-axis.

Together with the fixed coordinate system, we shall use the coordinate system $O\xi_1\eta_1\zeta$ rotating around the ζ-axis with an angular velocity ω. Denoting the absolute velocity vector of an arbitrary particle of the rigid body or the fluid by $\mathbf{v}(v_1, v_2, v_3)$, its position vector by $\mathbf{r}(\xi, \eta, \zeta)$, and its velocity vector with respect to the coordinate system $O\xi_1\eta_1\zeta$, by $\mathbf{u}(u, v, w)$, we have the relation

$$\mathbf{v} = \mathbf{u} + \boldsymbol{\omega} \times \mathbf{r}$$

The kinetic energy and the projection of the angular momentum on the ζ-axis can be expressed in the form

$$T = T_1 + \omega G_\zeta^1 + \tfrac{1}{2}\omega^2 S, \qquad G_\zeta = G_\zeta^1 + \omega S \tag{1.3}$$

Here,

$$T_1 = \frac{1}{2}\sum_\nu m_\nu (u_\nu^2 + v_\nu^2 + w_\nu^2), \qquad G_\zeta^1 = \sum_\nu m_\nu (\xi_\nu v_\nu - \eta_\nu u_\nu)$$

denote the kinetic energy and the projection of the angular momentum of the motion with respect to the coordinate system $\xi_1\eta_1\zeta$, and

$$S = \sum_\nu m_\nu (\xi_\nu^2 + \eta_\nu^2)$$

is the moment of inertia of the system with respect to the ζ-axis. In all the above formulas, the summation is performed over all the particles of the system, whose masses are denoted by m_ν ($\nu = 1, 2, \ldots$).

The angular velocity ω of the rotation of the coordinate system $\xi_1\eta_1\zeta$ may be prescribed arbitrarily. We shall assume it in such a way that $G_\zeta^1 = 0$, i.e. the projection of the angular momentum of the relative motion on the ζ-axis is equal to zero at any time [2]. This is equivalent, in view of (1.2) and (1.3), to the equation

$$\omega S = k \tag{1.4}$$

With this value of the angular velocity ω of the rotation of the coordinate system $\xi_1\eta_1\zeta$, the integral of energy (1.1) can be written in the form

$$T_1 + \frac{1}{2}\frac{k^2}{S} + V = h \tag{1.5}$$

Note. The form (1.5) of the integral of energy can be obtained also in a different way by generalizing Routh's method of cyclic coordinates in the dynamics of systems with finite number of degrees of freedom [4].

With the above assumptions concerning the constraints imposed on the rigid body and the acting forces, the system being investigated may actually perform uniform rotation around the fixed ζ-axis as one rigid body. In this, the system is in equilibrium with respect to the coordinate system $\xi_1\eta_1\zeta$ which rotates around the ζ-axis with the angular velocity ω_0 of the uniform rotation of the system.

In fact, the general equations of dynamics, expressing the d'Alembert-Lagrange principle, are valid for the actual motion of the system

$$\sum_\nu \{(m_\nu\xi''_\nu - F_{1\nu})\,\delta\xi_\nu + (m_\nu\eta''_\nu - F_{2\nu})\,\delta\eta_\nu + (m_\nu\zeta''_\nu - F_{3\nu})\,\delta\zeta_\nu\} = 0 \tag{1.6}$$

Here, ξ_ν'', η_ν'', ζ_ν'' are the components of the acceleration of the νth particle of the system, $F_{1\nu}$, $F_{2\nu}$, $F_{3\nu}$ are the components of the given forces, and $\delta\xi_\nu$, $\delta\eta_\nu$, $\delta\zeta_\nu$ are the components of the virtual displacement vector $\delta\mathbf{r}_\nu$.

In the case of uniform rotation of the whole system as one rigid body around the ζ-axis with the angular velocity ω_0, we have

$$\xi''_\nu = -\omega_0^2\xi_\nu, \qquad \eta''_\nu = -\omega_0^2\eta_\nu, \qquad \zeta''_\nu = 0$$

and, consequently, equation (1.6) assumes the form

$$\frac{1}{2}\omega_0^2\delta\sum_\nu m_\nu(\xi_\nu^2 + \eta_\nu^2) + \sum_\nu(F_{1\nu}\delta\xi_\nu + F_{2\nu}\delta\eta_\nu + F_{3\nu}\delta\zeta_\nu) = 0$$

or

$$\delta U = 0 \tag{1.7}$$

where the notation used is

$$U = \frac{1}{2}\omega_0^2 S - V \tag{1.8}$$

and the symbol δU denotes the variation of the expression U corresponding to virtual displacements of the system which are compatible with the constraints and do not change the volume of the fluid.

The expression (1.8) may be considered as the force-function of the given forces and centrifugal inertia forces. According to the principle of virtual work, equation (1.7) represents the condition of equilibrium of the system with respect to the coordinates $O\xi_1\eta_1\zeta$ if the latter rotate with the uniform angular velocity ω_0.

We introduce into consideration the function

$$W = \frac{1}{2}\frac{k_0^2}{S} + V \tag{1.9}$$

where k_0 is the value of the constant k for the case of uniform rotation of the whole system as one rigid body with the angular velocity ω_0 about the ζ-axis. The variation of this function corresponding to virtual displacements of the system is

$$\delta W = -\frac{1}{2}\frac{k_0^2}{S_0^2}\delta S + \delta V$$

where S_0 is the value of S for the steady motion.

Examining δW and $\delta U = 1/2\,\omega_0^2\delta S - \delta V$, and taking into account that $\omega_0 S_0 = k_0$, we conclude that equation (1.7) is equivalent to the equation

$$\delta W = 0 \tag{1.10}$$

Therefore, in the case of steady rotation of the system, the expression (1.9) has an extremal (stationary) value.

According to the definition (1.9), the function W depends on the coordinates of the body $q_1, \ldots, q_{n-1}$ (on which S and V also depend), on the form of the fluid, and on the value of the constant k_0. In the case of complete filling of the cavity in the body, the expression W is a function of $q_1, \ldots, q_{n-1}, k_0$.

The quantity k_0 may be considered as a variable parameter, and the results of the general theory of "equilibrium" of material systems with potential energy depending on parameters [5] can be used.

The condition (1.10) leads, as it is easy to see, to the equations

$$\frac{\partial W}{\partial q_i} = -\frac{1}{2}\omega_0^2 \frac{\partial S}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0 \qquad (i = 1, \ldots, n-1) \qquad (1.11)$$

for the coordinates q_i $(i = 1, \ldots, n - 1)$ of the rigid body in stationary motion, and to the equations for the pressure in the fluid. From these last equations we obtain the equation

$$\frac{1}{2}\omega_0^2 (\xi^2 + \eta^2) + U_2 = \text{const} \qquad (1.12)$$

for the free surface of the fluid which does not fill the cavity completely. Here, $U_2(\xi,\eta,\zeta)$ denotes the force-function of the body forces (per unit mass) acting on the fluid, such that the potential energy of the fluid is

$$V_2 = -\rho \int_\tau U_2 d\tau \quad (\tau \text{ is volume of the fluid})$$

For a constant value of the parameter k_0, equations (1.11) and (1.12) determine the coordinates of the rigid body and the form of the free surface of the fluid in steady motion. Under continuous change of the parameter, the real roots of equations (1.11) will change, i.e.

$$q_i^{(s)} = \varphi_i^{(s)}(k_0)$$

together with the corresponding forms of relative equilibrium of the fluid.

In the n-dimensional space $(q_1, \ldots, q_{n-1}, k_0)$, this last system of equations determines a real curve whose points correspond to different stationary motions. Separate branches of this curve intersect at the bifurcation points [5], at which at least two real roots of equations (1.11) coincide.

2. We shall consider a certain stationary motion of the system, corresponding to a given value of the constant k_0. Without loss of generality, we assume that the roots of equations (1.11) are $q_i = 0$ ($i = 1, \ldots, n-1$) for the given value k_0. Thus the fluid has the form of relative equilibrium F_0, determined by the free surface σ_0, given by equation (1.12) and the walls of the cavity.

Using the integral of energy (1.5), we shall investigate the stability of this steady motion of the rigid body with the cavity filled with fluid.

The mechanical system being investigated has $n + \infty$ degrees of freedom and it is necessary to define the concept of stability of its motion.

In the case of complete filling, as the stability of motion we shall assume the stability in the sense of Liapunov [5] with respect to the non-cyclic coordinates $q_1, \ldots, q_{n-1}$ (on which the potential energy V and the moment of inertia S of the system depend explicitly), generalized velocities $q'_1, \ldots, q'_{n-1}$ of the body, and the kinetic energy $T_1^{(2)}$ of the fluid.

In the case of partial filling, when the fluid in the cavity has a free surface, the problem becomes more complicated. As it has been explained by Liapunov [1] for fluids in general, the integral of energy is insufficient to indicate the character of the perturbed motion (caused by perturbations of its state of absolute or relative equilibrium) which corresponds to a stable motion in the mechanics of systems with a finite number of degrees of freedom. Liapunov has shown that the difficulty may be removed if the stable state of equilibrium is defined as the state which, after being subjected to sufficiently small perturbations, remains arbitrarily close to the original state. This proves to be adequate, at least until string or leaf shaped projections form on the surface of the fluid. Such projections may be large in linear dimensions, but their volume is small and, therefore, they carry small amounts of energy.

We shall assume the above definition and, following Liapunov, shall formulate certain pertinent concepts applicable to our problem. We shall compare the form F_0 of the relative equilibrium and the form F at an arbitrary instant of the perturbed motion; the motion of the particles of the fluid will not be considered, but we shall take into account the value of the kinetic energy of the fluid. The form F is determined by the free surface σ and the walls of the cavity which, at a given instant, are in contact with the fluid. If the perturbed motion is sufficiently close to the unperturbed motion, in the coordinate system xyz connected with the rigid body, the forms F_0 and F differ only in the free surfaces σ_0 and σ. Since the fluid is incompressible, the volume of the form F is obviously equal to the volume of the form F_0.

Let us consider a point P of the surface σ and the point P_0, nearest to P, of the surface σ_0. With changes of the position of the point P and, consequently, the position of the point P_0, the distance PP_0 changes; for certain positions of the point P it assumes its maximum value at a given instant of time. This maximum value has been called "the separation" by Liapunov. We shall denote this quantity by l. We shall use also the total deviation Δ of the form F from the form F_0 which is defined as the volume of the part of the form F which is within the form F_0 or, equivalently, the volume of the part of the form F_0 which is within the form F.

It is obvious that if the separation has a given value l, the deviation Δ has a certain maximum value which can be expressed as $l\psi(l)$, where $\psi(l)$ is a positive function having a certain definite upper bound. If l does not exceed an arbitrary number A, the function $\psi(l)$ has a minimum value which is different from zero. The minimum value of the deviation Δ for a given value of l is always equal to zero [2].

During a continuous motion of the body and the fluid, the separation l and the deviation Δ are, obviously, continuous functions of time.

We shall introduce now the following definition of stability of the motion of the system in the case of partial filling. Suppose that certain initial perturbations are applied to the system and we consider the subsequent perturbed motion. The considered motion of the system is stable if the initial value of the separation, the initial relative velocities of the particles of the fluid, and also the initial perturbations of the coordinates and velocities of the body can be selected sufficiently small in order to make the absolute values of the coordinates q_i, the velocities q_i', the kinetic energy $T_1^{(2)}$ of the fluid, and the separation l smaller than certain given arbitrarily small limits, for any time, or at least until the deviation becomes smaller than certain given, arbitrarily small, values. In the opposite case the motion of the system is unstable.

Thus, the unperturbed motion of the system is stable if for arbitrary positive numbers L_1 and L_2 (which may be arbitrarily small), it is possible to find a positive number λ such that for all the initial values of the coordinates q_{i0} and the generalized velocities q_{i0}' ($i = 1, \ldots, n - 1$) of the body, of the separation l_0, of the deviation Δ_0, and of the relative velocities of the fluid u_0, v_0, w_0, satisfying the conditions

$$|q_{i0}| \leqslant \lambda, \quad |q_{i0}'| \leqslant \lambda, \quad |l_0| \leqslant \lambda, \quad |u_0| \leqslant \lambda, \quad |v_0| \leqslant \lambda$$
$$|w_0| \leqslant \lambda, \quad \Delta_0 > \varepsilon l_0 \tag{2.1}$$

and for any $t \geqslant t_0$, or at least up to the time when

$$\Delta > \varepsilon l \tag{2.2}$$

the following inequalities are satisfied

$$|q_i| < L_1, \qquad |l| < L_1, \qquad |q_i'| < L_2, \qquad |T_1^{(2)}| < L_2 \tag{2.3}$$

Here, ε denotes a positive number smaller than the minimum value of the function $\psi(l)$ for $|l| \leqslant L_1$; the quantity εl may be considered as an admissible deviation of the fluid. We note that in the case of complete filling of the cavity with the fluid, the condition related to (2.2) should be omitted.

In the following discussion we shall encounter the concept of minimum of the expression W. If W is a function $W(q_1, \ldots, q_{n-1}, k_0)$, the minimum of this function for a fixed value of the parameter k_0 will be meant as the isolated minimum with respect to the variables $q_1, \ldots, q_{n-1}$ which are its explicit arguments. In the case of partial filling we shall assume the following definition, due to Liapunov [1], of the isolated minimum of W.

If W_0 is a minimum of the expression W for the steady motion being considered with $q_i = 0$ $(i = 1, \ldots, n-1)$, $l = 0$, $\Delta = 0$, then there exists a sufficiently small positive number E such that for all the values of the coordinates of the body q_i $(i = 1, \ldots, n-1)$ the separation l, and the deviation Δ, satisfying the conditions

$$|q_i| \leqslant E, \qquad |l| \leqslant E, \qquad \Delta > \varepsilon l$$

(where ε is a positive number, smaller than the minimum value of the function $\psi(l)$ for $|l| \leqslant E$), all the values of the difference $W - W_0$ are positive, and equal to zero only if $q_i = 0$ $(i = 1, \ldots, n-1)$, $l = 0$, $\Delta = 0$. We note that for any given value of l the difference $W - W_0$ may assume arbitrarily small values if the position of the body and the form of the fluid correspond to the values $|q_i|$ and Δ which are sufficiently small. But the limiting case where the condition $l \neq 0$ results in $q_i = 0$ $(i = 1, \ldots, n-1)$, $\Delta = 0$ (so that $W - W_0$ is also zero) is impossible if only those forms that can be taken by the fluid are considered. In order to remove this inconvenience the condition $\Delta > \varepsilon l$ has been introduced.

Theorem 2.1. If for a steady motion of the rigid body with the cavity filled with fluid the expression

$$W = \frac{1}{2}\frac{k_0^2}{S} + V$$

has an isolated minimum W_0, then the motion is stable.

Proof [2]. We shall perturb the steady motion of the system considered by assigning to its points certain sufficiently small initial displacements and velocities. Without external actions, the system will move accordingly to the integral of energy (1.5), which may be written in the form

$$T_1 + W + \frac{1}{2}\frac{k^2 - k_0^2}{S} = T_1^{(0)} + W^{(0)} + \frac{1}{2}\frac{k^2 - k_0^2}{S^{(0)}} \tag{2.4}$$

where the superscript (0) denotes the initial value of the corresponding quantity, and k is the area constant of the perturbed motion.

Let A be an arbitrarily small positive number not exceeding the number L_1, which will always be assumed smaller than the number E introduced above. We denote the smallest value of the expression W by W_1 if the separation l or one of the coordinates q_i ($i = 1, \ldots, n - 1$) has its absolute value equal to A, while the remaining quantities and the deviation Δ satisfy the conditions

$$|q_i| \leqslant A, \quad |l| \leqslant A, \quad \Delta \geqslant \varepsilon l$$

Since, according to the assumption, the expression W has the minimum W_0 for the steady motion, we have the inequality

$$W_1 > W_0$$

If, however, l and $|q_i|$ are sufficiently small and $\Delta > \varepsilon l$, the difference $|W - W_0|$ becomes arbitrarily small. We assume A small enough to have the inequality

$$|W_1 - W_0| < L_2 \tag{2.5}$$

satisfied.

The initial values of the coordinates q_i and the separation l can be selected such that the initial value of the expression W be smaller than the value W_1

$$W^{(0)} < W_1 \tag{2.6}$$

With this selection of the initial state of the system, we shall assume that the initial values of the coordinates q_i and the initial form of the fluid satisfy the inequalities

$$|q_i| < A, \qquad |l| < A, \qquad \Delta > \varepsilon l$$

For an arbitrary intial position and form of the fluid, the initial velocities can be chosen in such a way that the constant quantities

$$\frac{1}{2}|k^2 - k_0^2| \qquad T_1^{(0)}$$

be arbitrarily small. We take the values of these constants for which

$$\frac{1}{2}(k^2-k_0^2)\left(\frac{1}{S^{(0)}}-\frac{1}{S}\right)+T_1^{(0)}+W^{(0)}<W_1 \tag{2.7}$$

for any values which may be assumed by S if the conditions are satisfied

$$|q_i|\leqslant A, \qquad |l|\leqslant A \tag{2.8}$$

Considering the quantity λ, which appears in the definition of stability and which determines the region of initial perturbations, we shall assume its value in such a way that, fulfilling the conditions (2.1), we can satisfy the inequality (2.7) for all the values of S under the conditions (2.8). With this choice of the initial conditions we have, according to the energy integral, the following relation

$$T_1+W<W_1 \tag{2.9}$$

for $t \geqslant t_0$, as long as the conditions (2.8) are satisfied.

This implies that $W < W_1$, at least as long as $|q_i|$ and l do not exceed A. Since the initial values of the coordinates q_i and the separation l are, by assumption, smaller than A, with the initial deviation $\Delta > \varepsilon l$, and because q_i, l and Δ vary continuously in time, the values $|q_i|$ and $|l|$ cannot become larger than A without being previously equal to A. But the equalities

$$|q_i|=A \quad (i=1,\ldots,n-1), \qquad |l|=A$$

are, on the basis of (2.9) with $\Delta > \varepsilon l$, obviously impossible.

The inequality (2.9), with (2.5) taken into account, implies that $|T_1| < L_2$. We therefore conclude that

$$|q'_i|<L_2 \quad (i=1,\ldots,n-1), \qquad |T_1^{(2)}|<L_2$$

Consequently, if the motion of the system progresses continuously, i.e. q_i, l and Δ vary continuously in time, we have from the initial instant of time

$$|q_i|<L_1, \quad |q'_i|<L_2, \quad |l|<L_1, \quad |T_1^{(2)}|<L_2, \quad \Delta>\varepsilon l$$

These inequalities hold as long as the last of them is true. The theorem is thus proved.

Let us note that in the case of complete filling the conditions for l and Δ are superfluous and under the assumptions of the theorem we have

$$|q_i|<L_1, \quad |q_i'|<L_2 \quad (i=1,\ldots,n-1), \qquad |T_1^{(2)}|<L_2$$

for any time $t \geqslant t_0$.

Note 2.1. Liapunov [2] noted that in order to characterize the differences between the perturbed and unperturbed forms of the fluid, instead of l, we may introduce certain other quantities which become equal to zero only for the unperturbed form. For example, the total deviation Δ may be chosen as such a quantity. Then, in a similar manner as above, we can prove that if W has a minimum W_0 for a steady motion of the body with the cavity partially filled with fluid, for sufficiently small initial perturbations, $|q_i|$, $|q_i'|$ $(i = 1, \ldots, n-1)$, Δ and $T_1^{(2)}$ do not exceed arbitrarily small values at any time prescribed. The minimum W_0 should be understood in the sense that $W - W_0 > 0$ for all the values $|q_i|$ and Δ which do not vanish simultaneously and are smaller than certain constant limits.

Note 2.2. If the theorem of kinetic energies and the theorem of areas hold for the motion of the rigid body with the cavity filled with fluid with respect to the mass center of the total system, then Theorem 1.1 is valid also for this motion.

In this sense, Theorem 1.1 represents a generalization of the theorem of Liapunov concerning the stability of the equilibrium configurations of a rotating homogeneous fluid whose particles attract each other according to Newton's law.

In fact, if the system consists only of a gravitating fluid, it is

$$W = \frac{1}{2}\frac{k_0^2}{S} - \frac{f}{2}\iint \frac{d\tau d\tau'}{r}$$

and if for an equilibrium configuration the expression $\Pi = W/\pi f$ has its minimum, then this form of equilibrium is stable [1, 2].

Conclusion. If for a state of equilibrium of a rigid body with a cavity filled with fluid (for $k_0 = 0$), the potential energy of the system V has an isolated minimum V_0, then this state of equilibrium is stable [6].

We note that this conclusion is valid also in the case of relative equilibrium of a rigid body with a cavity filled with fluid.

Let us assume, for definiteness, that in addition to the forces derived from the force function V, nonconservative forces exist also and they are reducible to the moment N along the ζ-axis. The magnitude of this moment is such that the angular velocity ω of rotation of the rigid body around the ζ-axis remains constant at any time. In this case, instead of the integrals of energy (1.1) and the areas (1.2), we have the equations [4]

$$d(T + V) = N\omega dt, \qquad \frac{dG_\zeta}{dt} = N$$

from which we obtain

$$T + V - \omega G_\zeta = \text{const}$$

Introducing again into considerations the moving coordinate system $\xi_1\eta_1\zeta$ and recalling the relations (1.3), we can represent the energy condition in the form

$$T_1 + V - \tfrac{1}{2}\omega^2 S = \text{const}$$

The states of relative equilibrium of the rigid body with the fluid are determined by the condition (1.7). Repeating almost literally the proof of Theorem 2.1, we easily show the validity of the following theorem.

Theorem 2.2. If in a state of relative equilibrium of the rigid body with the cavity containing a fluid the expression

$$W_* = V - \tfrac{1}{2}\omega^2 S$$

has an isolated minimum, then this state of relative equilibrium is stable.

As we have already noted, if the equality (1.4) is fulfilled, the condition (1.7) is equivalent to equation (1.10). This makes possible the construction of the state of relative equilibrium for a constant angular velocity ω at steady motions with the existence of the integral of areas (1.2). It is easy to see that if the expression W_* has a minimum for a state of relative equilibrium, then the expression W has a minimum for the corresponding steady motion [4] also.

In fact, let W_{*0} be a minimum of the expression W_*, i.e. in a sufficiently small vicinity of the state of relative equilibrium

$$V - V_0 - \frac{1}{2}\omega^2 (S - S_0) > 0$$

and let us assume that for the corresponding steady motion W does not have a minimum, i.e. in a sufficiently small vicinity points exist for which

$$\frac{1}{2} k_0^2 \left(\frac{1}{S_0} - \frac{1}{S}\right) - V + V_0 \geqslant 0$$

Substituting $S_0\omega$ for k_0 in the last inequality and taking into account the preceding inequality, we obtain

$$-\frac{1}{2}\frac{\omega^2}{S}(S - S_0)^2 > 0$$

which is impossible. Consequently, if the state of relative equilibrium of the rigid body with the fluid is stable for ω = const, then the corresponding steady motion is stable for G_ζ = const also.

Let the expression W have a minimum for a given value of the parameter k_0, i.e. let the steady motion be stable. If we now continuously change the value of the parameter k_0, the roots of equations (1.11) will trace a branch C of the "equilibrium" curve. If the expression W varies also continuously, then for all the points of the curve C, for which W maintains its minimum value, the steady motions will be stable. The change of stability on this branch may occur only at the bifurcation points [5].

3. In the preceding considerations we have assumed that the fluid in the cavities of the body was nonviscous. We shall investigate now the motion of a rigid body with a viscous fluid, whose coefficient of viscosity will be denoted by μ. The motion of an incompressible fluid is described by the Navier-Stokes equation

$$\frac{d\mathbf{v}}{dt} = \mathbf{F} - \frac{1}{\rho}\,\mathrm{grad}\, p + \nu\Delta\mathbf{v}, \qquad \mathrm{div}\,\mathbf{v} = 0$$

where $\nu = \mu/\rho$ is the kinematical coefficient of viscosity, ρ is the density, and p is the hydrodynamic pressure. We assume that on the free surface the stress vector is $\mathbf{p}_n = -p_0\mathbf{n}$ (with $\mathbf{n}$ being the unit vector normal to the free surface, p_0 = const), and that at the rigid walls the fluid does not move with respect to the rigid body [7].

Using the above equations and the boundary conditions for the fluid, as well as the equations of motion of the rigid body, it is easy to obtain the following equation for the rate of dissipation of energy

$$\frac{d}{dt}(T+V) = -\mu\int_\tau \Big\{2\Big[\Big(\frac{\partial v_1}{\partial \xi}\Big)^2 + \Big(\frac{\partial v_2}{\partial \eta}\Big)^2 + \Big(\frac{\partial v_3}{\partial \zeta}\Big)^2\Big] +$$
$$+ \Big(\frac{\partial v_3}{\partial \eta} + \frac{\partial v_2}{\partial \zeta}\Big)^2 + \Big(\frac{\partial v_1}{\partial \zeta} + \frac{\partial v_3}{\partial \xi}\Big)^2 + \Big(\frac{\partial v_2}{\partial \xi} + \frac{\partial v_1}{\partial \eta}\Big)^2\Big\}\, d\tau \qquad (3.1)$$

valid under the assumption of Section 1 concerning forces acting on the system and continuity of its motion.

It follows from equation (3.1) that the motion of a rigid body with a cavity containing fluid is not accompanied by dissipation of energy (due to viscosity) only if at every point of the fluid the following equations are satisfied

$$\frac{\partial v_1}{\partial \xi} = \frac{\partial v_2}{\partial \eta} = \frac{\partial v_3}{\partial \zeta} = 0, \qquad \frac{\partial v_3}{\partial \eta} + \frac{\partial v_2}{\partial \zeta} = \frac{\partial v_1}{\partial \zeta} + \frac{\partial v_3}{\partial \xi} = \frac{\partial v_2}{\partial \xi} + \frac{\partial v_1}{\partial \eta} = 0 \quad (3.2)$$

Equations (3.2) express the conditions that the line-elements in the fluid neither elongate nor contract [7]. This is possible only if the fluid moves like one rigid mass together with the rigid body which encloses it.

Assuming, as previously, that the dissipative forces do not act along the cyclic coordinate q_n and taking into account that viscous forces are internal forces, it is easy to establish the existence of the integral of areas (1.2) for the case of viscous fluid. Introducing, as in Section 1, the system of coordinates $O\eta_1\gamma_1\zeta$ rotating around the axis $O\zeta$, on the basis of equation (3.1), we obtain instead of equation (1.5) the following inequality

$$T_1 + W + \frac{1}{2}\frac{k^2 - k_0^2}{S} \leqslant T_1^{(0)} + W^{(0)} + \frac{1}{2}\frac{k^2 - k_0^2}{S^{(0)}} \tag{3.3}$$

Theorem 3.1. If for a uniform rotation of the rigid body with a cavity containing viscous fluid, the expression W has an isolated minimum, then this motion is stable, and any perturbed motion sufficiently close to the unperturbed rotation will approach, in the limit, the steady motion of the system as one rigid body, provided that the condition $\Delta > \varepsilon l$ is always satisfied.

Proof. In this case, instead of equation (2.4) we have the inequality (3.3), and in order to prove the stability of uniform rotation of the rigid body with viscous fluid it is only necessary to repeat the proof of Theorem 2.1. We shall prove the second part of Theorem 3.1.

Let us consider an arbitrary perturbed motion of the system which at the initial instant of time is sufficiently close to the unperturbed motion. Suppose that the inequality $\Delta > \varepsilon l$ holds as long as $|l|$ does not exceed L_1. In this case the perturbed motion will always be sufficiently close to the stable unperturbed motion. According to equation (3.1), the total mechanical energy of the system is being dissipated during the perturbed motion until the fluid and the rigid body start moving as one rigid mass. Under these circumstances, there are two possible conclusions: either the total mechanical energy continuously decreases and the system finally comes to rest, or the system approaches a uniform rotation as one rigid body which corresponds to the extremum of the expression $1/2(k^2/S) + V$. The first conclusion, with $G_\zeta \neq 0$, is contradictory to the existence of the integral of areas (1.2), and thus only the second conclusion remains valid [8]. The theorem is proved.

Note. In a similar way we can prove also that Theorem 2.2 is true for the case of viscous fluid if during the motion ω = const.

Theorem 3.2. If for an isolated steady motion of the rigid body with a cavity containing viscous fluid the expression W has no minimum, then this motion is unstable.

Proof. For the considered motion let the roots of equations (1.11) be $q_i = 0$ ($i = 1, \ldots, n-1$) and $W_0 = 0$. We assume that there exists a sufficiently small positive number L_1 such that if all the coordinates q_i and the separation l satisfy the conditions

$$|q_i| \leqslant L_1, \qquad |l| \leqslant L_1 \tag{3.4}$$

the expression W has no extremum except at the point $q_i = 0$ ($i = 1, \ldots, n-1$), $l = 0$. This assumption implies that the investigated steady motion proves to be isolated. Since W does not have a minimum for this motion, within the region (3.4) another region exists where $W < 0$. Thus, in a region of small absolute values of the coordinates q_i, the separation l, and the relative velocities q_i', u, v, w, we can find – under our assumptions – the region of arbitrarily small values of the coordinates and velocities for which

$$T_1 + W < 0$$

We select the initial perturbations from this region in such a way that the area constant k remains equal to k_0. For $t \geqslant t_0$, the system moves according to the relation (3.3), which in present conditions takes the form

$$T_1 + W \leqslant T_1^{(0)} + W^{(0)} < 0$$

Suppose, contradicting the proposition, that the motion is stable. This means, according to the definition, that at any time, or at least as long as $\Delta > \varepsilon l$, the conditions (2.3) are satisfied.

If these conditions are satisfied, it is obviously possible to find a positive number L, depending on L_1 and L_2, which gives the upper bound for the absolute value of the mechanical energy of the system

$$|T_1 + W| < L \tag{3.5}$$

But in the region determined by the inequalities (2.3), equations (3.2) are never identically satisfied, except for the unperturbed motion being investigated. Consequently, the energy of the system will be continually dissipated and it will increasingly differ from its initial value. Finally, the absolute value of the energy will exceed L, and this contradicts the condition (3.5). Therefore, the system will move beyond the region (2.3). The theorem is thus proved.

4. We shall consider now the problem of stability of steady motion of a rigid body, with a cavity filled completely with fluid, attracted according to Newton's Law by a fixed center.

The center of attraction O will be assumed as the origin of the fixed coordinates $O\xi\eta\zeta$, while the mass center O_1 of the body with the cavity containing fluid, will be assumed as the origin of the moving axes x, y, z

which coincide with the principal axes of inertia of the system. The position of the system is determined by the coordinates ξ, η, ζ of the mass center O_1, and the three Eulerian angles. Introducing spherical coordinates R, ψ, φ of the mass center, we have

$$\xi = R\cos\psi\cos\varphi, \qquad \eta = R\cos\psi\sin\varphi, \qquad \zeta = R\sin\psi \tag{4.1}$$

The potential energy of the attractive forces can be represented with sufficient accuracy in the form [9]

$$V = -f\frac{M}{R} + \frac{3}{2}\frac{f}{R^3}\left(A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2 - \frac{A+B+C}{3}\right) \tag{4.2}$$

Here, f is the gravitation constant; M is the mass of the system; A, B, C are the principal central moments of inertia of the system; γ_1, γ_2, γ_3 are the directional cosines of the line OO_1, with respect to the axes x, y, z. The moment of inertia of the system with respect to the axis $O\zeta$ is

$$S = MR^2\cos^2\psi + A\beta_1^2 + B\beta_2^2 + C\beta_3^2 \tag{4.3}$$

where β_1, β_2, β_3 are the directional cosines of the axis $O\zeta$ with respect to the axes x, y, z.

The quantities β_i and γ_i are connected by the obvious relations

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \qquad \beta_1^2 + \beta_2^2 + \beta_3^2 = 1$$

Eliminating β_2 and γ_3 from (4.2) and (4.3) by the use of the above relations we obtain

$$V = -f\frac{M}{R} + \frac{3}{2}\frac{f}{R^3}\left[(A-C)\gamma_1^2 + (B-C)\gamma_2^2 - \frac{A+B-2C}{3}\right]$$
$$S = MR^2\cos^2\psi + B + (A-B)\beta_1^2 + (C-B)\beta_3^2 \tag{4.4}$$

Equations (1.11) are, in this case, satisfied by the following values of the variables [9]

$$R = R_0, \qquad \psi = 0, \qquad \beta_1 = \beta_3 = 0, \qquad \gamma_1 = \gamma_2 = 0 \tag{4.5}$$

with the constant R_0 satisfying the equation

$$\frac{k_0^2}{S_0^2}MR_0 = \frac{fM}{R_0^2} + \frac{3}{2}\frac{f}{R_0^4}(A+B-2C) \tag{4.6}$$

where

$$S_0 = MR_0^2 + B, \qquad k_0 = S_0\omega_0$$

The particular solution (4.5) describes the motion of the mass center of the system O_1 along a circular orbit of the radius R_0 situated in the plane $O\xi\eta$, with the angular velocity ω_0; the axis O_1z coincides with the line OO_1, the axis O_1x is tangent to the orbit, and the axis O_1y is parallel to the axis $O\zeta$. This motion will be assumed as the unperturbed motion, and we shall investigate its stability.

For this purpose, it is necessary to find the conditions for a minimum of the function W corresponding to the motion (4.5). With (4.5) and (4.6), we have

$$\frac{\partial^2 W}{\partial R^2} = \frac{f}{R_0^3}\left[M\left(1 - 4\frac{B}{S_0}\right) - \frac{3}{2}\frac{A+B-2C}{R_0^2}\left(1 + 4\frac{B}{S_0}\right)\right]$$

$$\frac{\partial^2 W}{\partial \psi^2} = MR_0^2\omega_0^2, \qquad \frac{\partial^2 W}{\partial \beta_1^2} = (B - A)\,\omega_0^2, \qquad \frac{\partial^2 W}{\partial \beta_3^2} = (B - C)\,\omega_0^2$$

$$\frac{\partial^2 W}{\partial \gamma_1^2} = 3\frac{f}{R_0^3}(A - C), \qquad \frac{\partial^2 W}{\partial \gamma_2^2} = 3\frac{f}{R_0^3}(B - C)$$

and all the remaining second derivatives of the function W equal identically to zero.

Thus, the conditions for a minimum of the function W reduce to the inequalities

$$B > A > C \tag{4.7}$$

which, according the Theorem 2.1, represent the sufficient conditions of stability of the unperturbed motion (4.5), of the rigid body with a cavity containing fluid, with respect to the variables [10]

$$R, \quad \psi, \quad \gamma_1, \quad \gamma_2, \quad \gamma_3, \quad \beta_1, \quad \beta_2, \quad \beta_3$$

In the case of a viscous fluid, with the conditions (4.7), the perturbed motion will damp out approaching the steady motion in the form of uniform rotation of the whole system around the vector of angular momentum.

5. As the second example, we shall consider the problem of stability of the rotation around the vertical axis of a heavy rigid body with a cavity containing fluid, whose one point O is fixed. The fixed axis ζ will be directed vertically upwards, and we shall introduce a moving coordinate system $Oxyz$ connected with the rigid body.

The potential energy V and the moment of inertia with respect to the ζ-axis are

$$V = Mg\left(x_0\gamma_1 + y_0\gamma_2 + z_0\sqrt{1 - \gamma_1^2 - \gamma_2^2}\right) \tag{5.1}$$

$$S = (A - C)\,\gamma_1^2 + (B - C)\,\gamma_2^2 + C - 2\,(D\gamma_2 + E\gamma_1)\sqrt{1 - \gamma_1^2 - \gamma_2^2} - 2F\gamma_1\gamma_2$$

where M, x_0, y_0, z_0, are the mass and the coordinates of the mass center of the system; g is the acceleration in the gravity field; A, B, C, D, E, F are the moments and products of inertia with respect to moving axes;

$\gamma_1, \gamma_2, \gamma_3 = \sqrt{(1-\gamma_1^2-\gamma_2^2)}$ are the direction cosines of the ζ-axis with respect to the axes x, y, z. Equations (1.11) are, in this case,

$$\frac{\partial W}{\partial \gamma_1} = -\omega_0^2\left[(A-C)\gamma_1 - E\sqrt{1-\gamma_1^2-\gamma_2^2} - F\gamma_2 + \frac{(D\gamma_2+E\gamma_1)\gamma_1}{\sqrt{1-\gamma_1^2-\gamma_2^2}}\right] + $$
$$+ Mg\left(x_0 - \frac{z_0\gamma_1}{\sqrt{1-\gamma_1^2-\gamma_2^2}}\right) = 0$$

$$\frac{\partial W}{\partial \gamma_2} = -\omega_0^2\left[(B-C)\gamma_2 - D\sqrt{1-\gamma_1^2-\gamma_2^2} - F\gamma_1 + \frac{(D\gamma_2+E\gamma_1)\gamma_2}{\sqrt{1-\gamma_1^2-\gamma_2^2}}\right] + $$
$$+ Mg\left(y_0 - \frac{z_0\gamma_2}{\sqrt{1-\gamma_1^2-\gamma_2^2}}\right) = 0$$

For any magnitude of the angular velocity ω_0, they are satisfied if

$$\gamma_1 = \gamma_2 = 0 \tag{5.2}$$

$$D = E = 0, \qquad x_0 = y_0 = 0 \tag{5.3}$$

i.e. if the axis of rotation z coincides with the vertical axis and is a principal central axis of inertia of the system.

We assume the above conditions, and we shall consider that the axes x and y also coincide with the remaining principal axes of inertia of the system passing through the point O.

With the conditions (5.2) we have

$$\frac{\partial^2 W}{\partial \gamma_1^2} = (C-A)\omega_0^2 - Mgz_0, \qquad \frac{\partial^2 W}{\partial \gamma_2^2} = (C-B)\omega_0^2 - Mgz_0, \qquad \frac{\partial^2 W}{\partial \gamma_1 \partial \gamma_2} = 0$$

In the case of complete filling of the cavity with fluid, the conditions for a minimum of the function $W(\gamma_1, \gamma_2, k_0)$ reduce to the following inequalities

$$(C-A)\omega_0^2 - Mgz_0 > 0, \qquad (C-B)\omega_0^2 - Mgz_0 > 0 \tag{5.4}$$

which, according the Theorems 2.1 and 3.1, are the sufficient conditions of stability of a heavy unsymmetrical top with a cavity filled with fluid [11].

If the fluid partially fills the cavity, then equation (1.12) of its free surface σ_0 is, in the investigated steady motion, that of a paraboloid of revolution

$$\frac{1}{2}\omega_0^2(x^2+y^2) - gz = \text{const} \tag{5.5}$$

The form F of the fluid in a perturbed motion can be produced by superimposing a layer of fluid of variable thickness χ on the free surface (5.5) of the unperturbed form F_0 [4]. Since the volume F is equal to the volume F_0, the condition should be satisfied

$$\int_{\sigma_0} \chi d\sigma = 0 \tag{5.6}$$

In the case of a stable motion of the top with the fluid, the quantity χ is of the same order as the quantities γ_1 and γ_2; therefore, neglecting small quantities of higher order, we have

$$W - W_0 = \frac{1}{2}\Big\{[(C-A)\,\omega_0^2 - Mgz_0]\,\gamma_1^2 + [(C-B)\,\omega_0^2 - Mgz_0]\,\gamma_2^2 +$$

$$+ 2\rho \int_{\sigma_0} (\omega_0^2 z + g)(x\gamma_1 + y\gamma_2)\,\chi d\sigma + \rho \int_{\sigma_0} \sqrt{\omega_0^4 (x^2+y^2) + g^2}\,\chi^2 d\sigma +$$

$$+ \frac{\omega_0^2}{S_0}\Big(\rho \int_{\sigma_0} (x^2+y^2)\,\chi d\sigma\Big)^2\Big\} + \cdots \tag{5.7}$$

In this, we have used the examples of the calculations of the integrals over the volume τ of the perturbed form F, developed in the theory of stability of the equilibrium configurations of rotating fluids [4].

The relation (5.7) indicates that the conditions (5.4) are necessary for the expression W to have a minimum for a steady motion of the top with a cavity partially filled with fluid [1].

Rough estimates of the sufficient conditions for a minimum of the expression W can be obtained by writing the right-hand side of equation (5.7) under the integral over the surface σ_0, and requiring that the integrand be positive-definite with respect to the variables γ_1, γ_2, χ.

The sufficient condition of stability of the top with a cavity partially filled with fluid, can be derived by expanding the quantity χ in a series of a complete system of eigenfunctions of the corresponding eigenvalue problem.

Let us assume, e.g. that the cavity has the form of a body of revolution whose sides are generated by rotating a convex plane curve around the z-axis, while the top and the bottom are planes

$$z = h - c, \qquad z = h + c$$

We assume, for simplicity, that the square of the angular velocity

$$\omega_0^2 \gg 2g\,(h + c) \tag{5.8}$$

and thus, the free surface (5.5) is not very different from the circular cylinder

$$x^2 + y^2 = a^2$$

Neglecting that small difference we can assume [12]

$$\chi = \sum_{k=0}^{\infty} (A \quad \cos\varphi + B_k \sin\varphi) \cos\frac{k\pi}{2c}(z - h + c) \tag{5.9}$$

and we obtain, according to the relation (5.7)

$$W - W_0 = \frac{1}{2}\Big\{[(C - A)\omega_0^2 - Mgz_0]\gamma_1^2 + [(C - B)\omega_0^2 - Mgz_0]\gamma_2^2 +$$

$$+ 4\rho\pi ca^2h\omega_0^2(A_0\gamma_1 + B_0\gamma_2) - 16\rho a^2\omega_0^2\frac{c}{\pi}\sum_{j=0}^{\infty}\frac{A_{2j+1}\gamma_1 + B_{2j+1}\gamma_2}{(2j+1)^2} +$$

$$+ \rho\pi a^2c\omega_0^2[2(A_0^2 + B_0^2) + \sum_{k=1}^{\infty}(A_k^2 + B_k^2)] + \cdots$$

It is easy to show that the right-hand side of this equality is positive-definite with respect to the variables γ_1, γ_2, A_k, B_k ($k = 0,1,2\ldots$) if the single condition is satisfied

$$\left(C - A - 2\rho\pi a^2c\frac{3h^2 + c^2}{3}\right)\omega_0^2 - Mgz_0 > 0 \tag{5.10}$$

with $A \geqslant B$.

According to Theorems 2.1 and 3.1, the equality (5.10) with the condition (5.8) is, in first approximation, the sufficient condition of stability of rotation around the vertical axis of a heavy top with a cavity partially filled with fluid. The quantities A, B, C, and z_0, in the inequality (5.10), should be calculated for the unperturbed position of the top and fluid.

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